**Chapter-4**

**Eigenvalues and Eigenvectors**

Let is a square matrix. A non-zero vector in is called an eigenvector of if is a scalar multiple of ; that is for some scalar . The scalar is called an eigenvalue of and is called the eigenvector of corresponding to.

**Example:** The vector is an eigenvector of corresponding to the eigenvalue .

.

**Characteristic matrix:**

Provided that is a square matrix of order . Then the matrix is called the characteristic matrix where is scalar and is the unit matrix.

**Example:**

Characteristic matrix is

**Characteristic polynomial:**

The determinant of characteristic matrix is the polynomial of . Characteristic polynomial is

.

**Characteristic equation:**

The equation is called characteristic equation for matrix .

For example, is characteristic equation for the above matrix .

**Characteristic root(s) or eigenvalue(s):**

The roots of the characteristic equation are called characteristic values or eigenvalues of matrix A.

So, the characteristic values or eigenvalues are 1,1 and 5

**Example:**

Find the eigenvalues and eigenvectors of the matrix

**Solution:**

The characteristic matrix of is,

The characteristic polynomial of matrix is

The characteristic equation of matrix is

So, the characteristic roots or the eigenvalues of matrix is

Now consider is an eigenvector of matrix corresponding to the eigenvalue . From the definition of eigenvalue and eigenvector

**MatLab** command for finding eigenvalues and eigenvectors:

>> A=[1 2 -1;0 -2 0;0 5 2];

>> [v,d]=eig(sym(A));

>> eigenvalues=eig(A)'

eigenvalues =

1 2 -2

>> eigenvectors=double(v)

eigenvectors =

1.0000 -1.0000 0.8667

0 0 -0.8000

0 1.0000 1.0000

So,

From the above equal matrices, we have

For , system of equations (1) becomes,

Solving we get

Hence is a free variable. Let, , where is any real number. Therefore, the eigenvector of corresponding to the eigenvalue are the non-zero vectors of the form . In particular, if , then is an eigenvector corresponding to the eigenvalue of

Again, for , system of equations (1) becomes,

This system has one free variable. Let

and

Therefore, .

In particular, let

So, is an eigenvector corresponding to the eigenvalue

For , system of equations (1) becomes,

Hence, and is free variable. Let then we have

Therefore,

In particular, if we have

Therefore, eigenvectors are , and corresponding to the eigenvalues respectively.

**Example:** Solve the following system of differential equation using eigenvalues and eigenvectors.

with.

where and .

**Solution:**

Let, and

So,

We write,

Now the system of differential equation can be written as

Let and be the eigenvalue and eigenvector of respectively and is an integral constant then we have the solution of the form,

The characteristic matrix of is

The characteristic polynomial of is

The characteristic equation of is

So the characteristic roots or the eigenvalues of is

Now, consider is the eigenvector of corresponding to the eigenvalue . From the definition of eigenvalue and eigenvector, .

So,

From the above equal matrices, we have

For , system of equations (1) becomes,

Solving the above system, we get

Here is a free variable. Let .

Thus, we get and.

Therefore, the eigenvector of corresponding to the eigenvalue are the non-zero vectors of the form .

Again, for , system of equations (1) becomes,

Solving the above system, we get

Here is a free variable. Let .

Thus, we get and.

Therefore, the eigenvector of corresponding to the eigenvalue are the non-zero vectors of the form .

So, the solution of the system of differential equation can be written as,

We can write,

Solving the system for and , we have, and . Now,

Therefore,

**Sample Exercise-4.1**

1. Find the eigenvalues and eigenvectors of the following matrices

|  |  |
| --- | --- |
|  | Ans: , , ,  , |
|  | Ans: , , ,  , |
|  | Ans: , , ,  , |
|  | Ans: , ,  , ,  , , |
|  | Ans: , ,  , ,  , , |

1. Solve the following system of differential equations using eigenvalue and eigenvector where and .

|  |  |
| --- | --- |
| with. | Ans. |
| with. | Ans. |
| with. | Ans. |

**Cayley-Hamilton Theorem:**

Every square matrix is a zero of its characteristic polynomial.

Or,

Every square matrix satisfies its characteristic equation

Example: Verify the Cayley-Hamilton theorem for the matrix

Solution: The characteristic matrix of is

Therefore, the characteristic equation of the matrix is

Now in order to verify Cayley –Hamilton theorem we have to show that

So,

Hence Cayley-Hamilton theorem is verified.

**Example:** Using Cayley-Hamilton theorem find the inverse of the matrix

**Solution:** The characteristic matrix of A is

Therefore, the characteristic equation of the matrix A is

Now according to the Cayley –Hamilton theorem, we have

Here,

.

**Example:** Using Cayley-Hamilton theorem find the inverse of the matrix

The characteristic matrix of is

.

Therefore, the characteristic equation of the matrix is

Now according to the Cayley –Hamilton theorem, we have

**Sample Exercise-4.2**

State the Cayley-Hamilton theorem. Hence find the inverse of the following matrices using Cayley –Hamilton theorem and verify your result.

|  |  |
| --- | --- |
|  | Ans. |
|  | Ans. |
|  | Ans. |
|  | Ans. |
|  | Ans. |

**Vector**

**Vector Spaces**

**Vectors in :**

The set of all ordered pairs of real numbers is called two-dimensional vector space and is denoted by .

**Example:**

**Vectors in :**

The set of all ordered triplets of real numbers is called three-dimensional vector space and is denoted by

**Vectors in :**

If is a positive integer then the set of all ordered ***n*** triplets of real numbers is called vector -space and is denoted by and if ; , then is called a -dimensional vector in . A particular ***n*** triplets in is called co-ordinates of point.

**Addition of two vectors in :**

If **a**and be two vectors in then

C:\Users\Administrator\Desktop\Smart draw Figures\vector.tif

Fig: addition of two vectors.

**Subtraction of two vectors in :**

If **a**and be two vectors in then

C:\Users\Administrator\Desktop\Smart draw Figures\vector3.tif

Fig: Subtraction of two vectors.

**Scalar multiplication of vectors in :**

Let be a scalar and where is a vector in then

**Zero Vector:**

The vector whose components are all zero is called the zero vector and is denoted by **0** and defined by

**Dot or Inner product of two non-zero vectors:**

Let and are two non-zero vectors and be the angle between. The dot product of and is denoted by and defined by

**Parallel vectors in :**

Let , then the vector is called parallel vector of if . If the non-zero scalar then they are in the same directed and if then they are in opposite directed.

**Perpendicular vectors in**

Let then and are said to be perpendicular (or orthogonal) if

**Example:** Let where

Hence and are perpendicular vector.

**Distance between two vectors in**

Let where and , then the distance between and , denoted by are defined by

**Example:** If then

**Norm or Length in :**

Let, where then the norm or length or modulus of denoted by and is defined by

**Example:** Let; where , then find

(iii)

**Solution:**

.

**Vector Spaces:**

Let be a field of scalars and be a non-empty set of vectors. If contains the following rules of vector addition and scalar multiplication, then is called vector space over .

**In vector addition:**

**In scalar multiplication:**

**Subspace:**

Let be a non empty subset of a vector space over the field. We call a subspace of if and only if is a vector space over the field under the laws of vector addition and scalar multiplication defined on , or equivalently, is a subspace of wherever and implies that .

**Example:** Show that is a subspace of the vector space **.**

**Solution:** Here

Hence is non empty set. Again, let and

and

For any scalars α, β we get

Now,

∴ . Hence is a subspace of  **.**

**Linear combination, Dependency and Independency of Vector**

**Linear combination:**

Let be a vector space, where and . If , then is called linear combination of .

**Linear dependency of vectors:**

Let be a vector space, where and . If , and at least one of the element of the set is not zero, then the vectors are linearly dependent.

**Linear independency of vectors:**

Let be a vector space and and . If , and all of the elements of the set are zero , then the vectors are linearly independent.

**Example:** Write the vector as a linear combination of the vectors

.

**Solution:** Let  **;** where are scalars.

Equating corresponding components

Solving using elementary row operation, we get .

∴

Hence is a linear combination of the vectors .

**Example:** Test whether the vectors are linearly dependent or independent.

**Solution:** Let  **;** where are scalars.

We can write from above equation

Solving the above linear system, we get .

Hence the vectors are linearly independent

**Sample Exercise-4.3**

Write the vector as a linear combination of the vectors  **,** where

1. ,**.**
2. ,**.**
3. ,**.**

Test whether the following vectors are linearly independent or dependent.

1. .